Ivy Tech
MATH 261 - Multivariate Calculus
Final Exam Review

December 2025

Problem 1.

Problem. The surface is given by

$$z = 3\big(x^2 + y^2\big),$$

and R is the rectangle in the xy–plane defined by $0 \le x \le 2$, $1 \le y \le 3$.

(a). Set up an iterated integral (with the order $dx\,dy$) for the volume of the solid that lies under the surface and above the region R.

(b). Evaluate the iterated integral to find the volume of the solid.

Problem 2.

Evaluate the integral

$$\iint_R \cos(x^2 + y^2)\,dA,$$

where R is the region in the first quadrant between the circles (centered at the origin) of radii 2 and 5.

Problem 3.

Evaluate the double integral

$$\iint_D y^2 e^{xy}\,dA,$$

where D is the region bounded by $y = x$, $y = 4$, and $x = 0$.

Problem 4.

Evaluate the integral

$$\iint_R \frac{x-2y}{3x-y}\, dA,$$

where R is the parallelogram enclosed by the lines

$$x-2y=1, \qquad x-2y=6, \qquad 3x-y=2, \qquad 3x-y=8.$$

(Note: The actual problem on the Final Exam, which is similar to this one, is divided into several parts so that you can work through it step by step. Here, however, I wrote the solution in a single step. The important thing is to understand the overall idea of how to solve the problem.)

Problem 5.

This problem is about matching the vector field with its plot. You can find a very similar problem in Review Test (Problem 6).

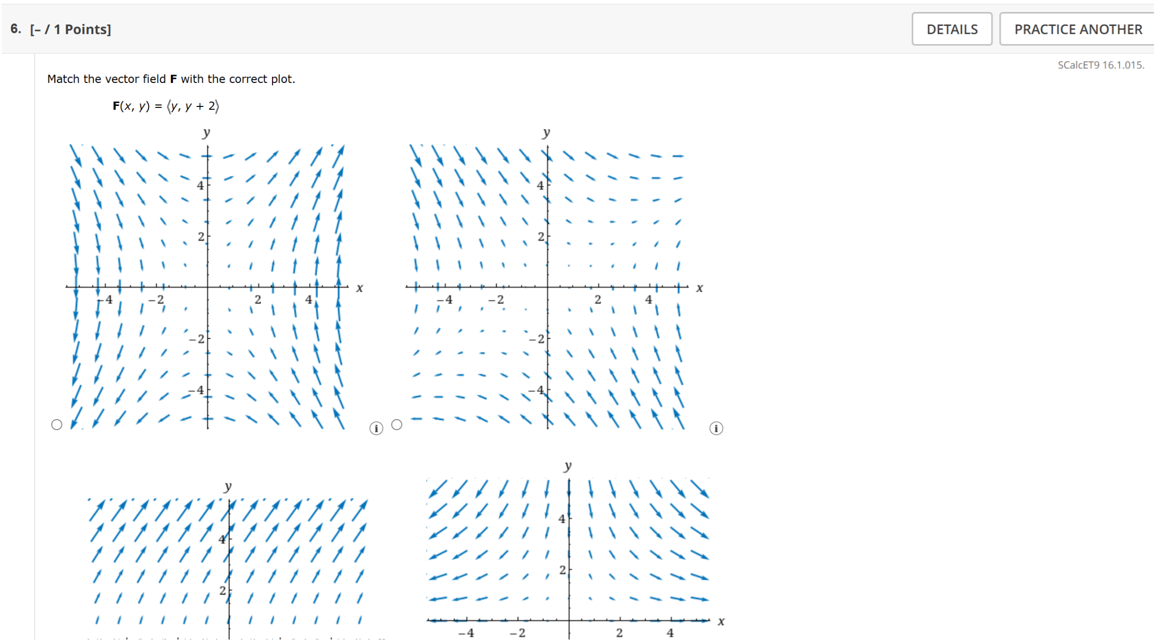

Figure 1: This is the Problem 6 in Review Test.

Problem 6.

Evaluate the line integral

$$\int_C \left(x^2+y^2+z^2\right) ds,$$

where C is the space curve given by

$$x=t, \qquad y=\cos(3t), \qquad z=\sin(3t), \qquad 0\le t\le 2\pi.$$

Problem 7.

Consider the vector field

$$\mathbf{F}(x, y) = \left(2 + 5xy^2\right)\mathbf{i} + 5x^2y\,\mathbf{j},$$

and let C be the arc of the hyperbola $y = \dfrac{1}{x}$ from $(1, 1)$ to $(3, \frac{1}{3})$.

(a). Find a function $f(x, y)$ such that $\mathbf{F} = \nabla f$.

(b). Use part (a) to evaluate the line integral $\displaystyle\int_C \mathbf{F} \cdot d\mathbf{r}$.

Problem 8.

A particle starts at the origin, moves along the x-axis to $(3, 0)$, then along the quarter–circle

$$x^2 + y^2 = 9, \qquad x \geq 0,\ y \geq 0,$$

to the point $(0, 3)$, and then down the y-axis back to the origin. Use Green's theorem to find the work done on this particle by the force field

$$\mathbf{F}(x, y) = \left(\sin x,\ \sin y + xy^2 + \tfrac{1}{3}x^3\right).$$

Problem 9.

Determine whether the vector field

$$\mathbf{F}(x, y, z) = \left(\ln(2y),\ \frac{x}{y} + \ln(5z),\ \frac{y}{z}\right)$$

is conservative on a suitable domain. If it is conservative, find a function f such that $\mathbf{F} = \nabla f$. (If not conservative, answer DNE.)

Problem 10.

Find the area of the surface

$$z = \frac{2}{3}\left(x^{3/2} + y^{3/2}\right), \qquad 0 \leq x \leq 2,\ 0 \leq y \leq 1.$$

Problem 11.

Use Stokes' theorem to evaluate

$$\oint_C \mathbf{F} \cdot d\mathbf{r},$$

where

$$\mathbf{F}(x, y, z) = \left(-yx^2,\ xy^2,\ e^{xy}\right),$$

and C is the circle in the xy-plane centered at the origin of radius 3, oriented counterclockwise as viewed from above.

Problem 12.

Use the Divergence Theorem to calculate the flux

$$\iint_S \mathbf{F} \cdot d\mathbf{S},$$

where

$$\mathbf{F}(x, y, z) = \left(2x^3 + 2y^3,\ 2y^3 + 2z^3,\ 2z^3 + 2x^3\right),$$

and S is the sphere centered at the origin of radius 3, oriented outward.

Answers.

Problem 1.

(a) The volume is

$$V = \iint_R 3(x^2+y^2)\,dA = \int_{y=1}^{3}\int_{x=0}^{2} 3\bigl(x^2+y^2\bigr)\,dx\,dy.$$

(b) Evaluate the inner integral first:

$$\int_{x=0}^{2} 3(x^2+y^2)\,dx = 3\left[\frac{x^3}{3}+y^2x\right]_0^2 = 3\Bigl(\frac{8}{3}+2y^2\Bigr) = 8+6y^2.$$

Now integrate with respect to y:

$$V = \int_1^3 \bigl(8+6y^2\bigr)\,dy = \Bigl[8y+2y^3\Bigr]_1^3 = (8{\cdot}3+2{\cdot}27)-(8{\cdot}1+2{\cdot}1) = (24+54)-(8+2) = 78-10 = 68.$$

Thus, the volume is $\boxed{68}$ (cubic units).

Problem 2.

Switch to polar coordinates: $x = r\cos\theta$, $y = r\sin\theta$, and $dA = r\,dr\,d\theta$. In the first quadrant θ runs from 0 to $\frac{\pi}{2}$, and r runs from 2 to 5. Thus

$$\iint_R \cos(x^2+y^2)\,dA = \int_{\theta=0}^{\pi/2}\int_{r=2}^{5} \cos(r^2)\,r\,dr\,d\theta.$$

Evaluate the inner integral by the substitution $u = r^2$, so $du = 2r\,dr$ and $r\,dr = \frac{1}{2}\,du$:

$$\int_{r=2}^{5} \cos(r^2)\,r\,dr = \frac{1}{2}\int_{u=4}^{25} \cos u\,du = \frac{1}{2}\bigl[\sin u\bigr]_4^{25} = \frac{1}{2}\bigl(\sin 25-\sin 4\bigr).$$

Now integrate with respect to θ:

$$\iint_R \cos(x^2+y^2)\,dA = \int_0^{\pi/2} \frac{1}{2}\bigl(\sin 25-\sin 4\bigr)\,d\theta = \frac{1}{2}\bigl(\sin 25-\sin 4\bigr)\cdot\frac{\pi}{2}.$$

Therefore, the value of the integral is

$$\boxed{\iint_R \cos(x^2+y^2)\,dA \;=\; \frac{\pi}{4}\,\bigl(\sin 25-\sin 4\bigr)}.$$

Problem 3.

The region D can be described as

$$D = \{(x, y) : 0 \le x \le y,\ 0 \le y \le 4\}.$$

(a) Iterated integrals in both orders.

Order $dx\,dy$: for fixed y we have $0 \le x \le y$, and y runs 0 to 4, so

$$\iint_D y^2 e^{xy}\, dA = \int_{y=0}^{4} \int_{x=0}^{y} y^2 e^{xy}\, dx\, dy.$$

Order $dy\,dx$: for fixed x we have $x \le y \le 4$, and x runs 0 to 4, so

$$\iint_D y^2 e^{xy}\, dA = \int_{x=0}^{4} \int_{y=x}^{4} y^2 e^{xy}\, dy\, dx.$$

(b) Evaluate using the easier order.

The inner integral with respect to x is easy (because $\partial/\partial x(e^{xy}) = ye^{xy}$):

$$\int_{x=0}^{y} y^2 e^{xy}\, dx = y^2 \left[\frac{1}{y} e^{xy}\right]_{x=0}^{x=y} = y\big(e^{y^2} - 1\big).$$

Thus

$$\iint_D y^2 e^{xy}\, dA = \int_0^4 y\big(e^{y^2} - 1\big)\, dy = \int_0^4 y e^{y^2}\, dy - \int_0^4 y\, dy.$$

Compute each piece. With $u = y^2$, $du = 2y\, dy$,

$$\int_0^4 y e^{y^2}\, dy = \frac{1}{2} \int_0^{16} e^u\, du = \frac{1}{2}\big(e^{16} - 1\big).$$

And

$$\int_0^4 y\, dy = \left.\frac{y^2}{2}\right|_0^4 = \frac{16}{2} = 8.$$

Therefore

$$\iint_D y^2 e^{xy}\, dA = \frac{1}{2}\big(e^{16} - 1\big) - 8 = \frac{1}{2} e^{16} - \frac{17}{2}.$$

$$\boxed{\iint_D y^2 e^{xy}\, dA = \frac{1}{2} e^{16} - \frac{17}{2}}.$$

Problem 4.

Make the change of variables

$$u = x - 2y, \qquad v = 3x - y.$$

Under this map the region R becomes the rectangle

$$S = \{(u, v)\colon 1 \le u \le 6,\ 2 \le v \le 8\}.$$

Compute the Jacobian. Writing

$$\begin{pmatrix} u \\ v \end{pmatrix} = \begin{pmatrix} 1 & -2 \\ 3 & -1 \end{pmatrix} \begin{pmatrix} x \\ y \end{pmatrix},$$

we have

$$\det \frac{\partial(u, v)}{\partial(x, y)} = \det \begin{pmatrix} 1 & -2 \\ 3 & -1 \end{pmatrix} = 1 \cdot (-1) - (-2) \cdot 3 = 5.$$

Hence

$$\frac{\partial(x, y)}{\partial(u, v)} = \frac{1}{5}.$$

Also note that the integrand simplifies:

$$\frac{x - 2y}{3x - y} = \frac{u}{v}.$$

Thus, the integral becomes

$$\iint_R \frac{x - 2y}{3x - y}\, dA = \iint_S \frac{u}{v} \left| \frac{\partial(x, y)}{\partial(u, v)} \right| du\, dv = \frac{1}{5} \int_{v=2}^{8} \int_{u=1}^{6} \frac{u}{v}\, du\, dv.$$

Evaluate the inner integral:

$$\int_{u=1}^{6} \frac{u}{v}\, du = \frac{1}{v} \int_1^6 u\, du = \frac{1}{v} \cdot \frac{6^2 - 1^2}{2} = \frac{1}{v} \cdot \frac{36 - 1}{2} = \frac{35}{2v}.$$

So

$$\iint_R \frac{x - 2y}{3x - y}\, dA = \frac{1}{5} \int_2^8 \frac{35}{2v}\, dv = \frac{35}{10} \int_2^8 \frac{1}{v}\, dv = \frac{7}{2} \Big[\ln v\Big]_2^8 = \frac{7}{2} \ln \frac{8}{2} = \frac{7}{2} \ln 4.$$

Therefore

$$\boxed{\iint_R \frac{x - 2y}{3x - y}\, dA = \frac{7}{2} \ln 4.}$$

Problem 5.

Check Review Test for Final Exam in Cengage for similar problem.

Problem 6.

Parametrize the curve by $\mathbf{r}(t) = (t, \cos(3t), \sin(3t))$ for $0 \le t \le 2\pi$. Then

$$\mathbf{r}'(t) = \big(1,\ -3\sin(3t),\ 3\cos(3t)\big).$$

The speed is

$$\|\mathbf{r}'(t)\| = \sqrt{1^2 + (-3\sin(3t))^2 + (3\cos(3t))^2} = \sqrt{1 + 9\sin^2(3t) + 9\cos^2(3t)} = \sqrt{1+9} = \sqrt{10}.$$

The integrand becomes

$$x^2 + y^2 + z^2 = t^2 + \cos^2(3t) + \sin^2(3t) = t^2 + 1.$$

Hence, the line integral is

$$\int_C \big(x^2 + y^2 + z^2\big)\, ds = \int_0^{2\pi} \big(t^2 + 1\big)\, \|\mathbf{r}'(t)\|\, dt = \sqrt{10} \int_0^{2\pi} \big(t^2 + 1\big)\, dt.$$

Evaluate the elementary integral:

$$\int_0^{2\pi} (t^2 + 1)\, dt = \left[\frac{t^3}{3} + t\right]_0^{2\pi} = \frac{(2\pi)^3}{3} + 2\pi = \frac{8\pi^3}{3} + 2\pi.$$

Therefore,

$$\boxed{\int_C \big(x^2 + y^2 + z^2\big)\, ds = \sqrt{10}\left(\frac{8\pi^3}{3} + 2\pi\right) = 2\pi\sqrt{10}\left(\frac{4\pi^2 + 3}{3}\right).}$$

Problem 7.

(a) We seek f with

$$f_x = 2 + 5xy^2, \qquad f_y = 5x^2 y.$$

Integrate f_x with respect to x:

$$f(x, y) = \int (2 + 5xy^2)\, dx = 2x + \frac{5}{2}x^2y^2 + g(y),$$

where $g(y)$ is an arbitrary function of y. Differentiate this expression with respect to y and match f_y:

$$f_y = \frac{5}{2} \cdot 2x^2 y + g'(y) = 5x^2 y + g'(y).$$

Since we must have $f_y = 5x^2y$, it follows that $g'(y) = 0$. Thus g is constant, and we may take it to be 0. One convenient potential is

$$\boxed{f(x, y) = 2x + \frac{5}{2}x^2y^2.}$$

(b) Because $\mathbf{F} = \nabla f$, the line integral is path independent and equals the difference of f at the endpoints. Evaluate f at the endpoints $(1,1)$ and $(3, \frac{1}{3})$:

$$f(3, \tfrac{1}{3}) = 2 \cdot 3 + \frac{5}{2} \cdot 3^2 \cdot \left(\tfrac{1}{3}\right)^2 = 6 + \frac{5}{2} \cdot 9 \cdot \frac{1}{9} = 6 + \frac{5}{2} = \frac{17}{2},$$

$$f(1,1) = 2 \cdot 1 + \frac{5}{2} \cdot 1^2 \cdot 1^2 = 2 + \frac{5}{2} = \frac{9}{2}.$$

Therefore,

$$\int_C \mathbf{F} \cdot d\mathbf{r} = f(3, \tfrac{1}{3}) - f(1,1) = \frac{17}{2} - \frac{9}{2} = 4.$$

$$\boxed{\int_C \mathbf{F} \cdot d\mathbf{r} = 4.}$$

Problem 8.

Let $\mathbf{F} = (P, Q)$ with

$$P(x,y) = \sin x, \qquad Q(x,y) = \sin y + xy^2 + \tfrac{1}{3}x^3.$$

The closed path C described in the problem is the positively oriented boundary of the quarter–disk

$$D = \{(x,y) : x^2 + y^2 \le 9,\ x \ge 0,\ y \ge 0\}.$$

By Green's theorem,

$$\oint_C \mathbf{F} \cdot d\mathbf{r} = \iint_D \left(\frac{\partial Q}{\partial x} - \frac{\partial P}{\partial y} \right) dA.$$

Compute the partial derivatives:

$$\frac{\partial Q}{\partial x} = y^2 + x^2, \qquad \frac{\partial P}{\partial y} = 0,$$

so the integrand is $x^2 + y^2$. Thus

$$\oint_C \mathbf{F} \cdot d\mathbf{r} = \iint_D (x^2 + y^2)\, dA.$$

Switch to polar coordinates $x = r\cos\theta,\ y = r\sin\theta$. Over the quarter–disk D we have $0 \le r \le 3$ and $0 \le \theta \le \frac{\pi}{2}$. Note that $x^2 + y^2 = r^2$ and $dA = r\,dr\,d\theta$. Therefore

$$\iint_D (x^2 + y^2)\, dA = \int_{\theta=0}^{\pi/2} \int_{r=0}^{3} r^2 \cdot r\, dr\, d\theta = \int_0^{\pi/2} \int_0^3 r^3\, dr\, d\theta.$$

Evaluate the radial integral:

$$\int_0^3 r^3\, dr = \left.\frac{r^4}{4}\right|_0^3 = \frac{3^4}{4} = \frac{81}{4}.$$

Now integrate over θ:

$$\iint_D (x^2+y^2)\,dA = \int_0^{\pi/2} \frac{81}{4}\,d\theta = \frac{81}{4}\cdot\frac{\pi}{2} = \frac{81\pi}{8}.$$

Hence the work done is

$$\boxed{\oint_C \mathbf{F}\cdot d\mathbf{r} = \frac{81\pi}{8}}.$$

Problem 9.

First note the natural domain for the field (where the logarithms are defined) is

$$D = \{(x,y,z) : y>0,\ z>0\},$$

which is open and simply connected.

Write $\mathbf{F} = (P,Q,R)$ with

$$P = \ln(2y), \qquad Q = \frac{x}{y} + \ln(5z), \qquad R = \frac{y}{z}.$$

Compute the mixed partials needed for the curl:

$$\frac{\partial P}{\partial y} = \frac{1}{y}, \qquad \frac{\partial Q}{\partial x} = \frac{1}{y},$$

$$\frac{\partial P}{\partial z} = 0, \qquad \frac{\partial R}{\partial x} = 0,$$

$$\frac{\partial Q}{\partial z} = \frac{1}{z}, \qquad \frac{\partial R}{\partial y} = \frac{1}{z}.$$

All corresponding mixed partials agree, so $\nabla\times\mathbf{F} = 0$ on D. Since D is simply connected, $\mathbf{F}$ is conservative on D.

To find a potential f, integrate P with respect to x:

$$f(x,y,z) = \int \ln(2y)\,dx = x\ln(2y) + g(y,z),$$

where g is an unknown function of y,z. Differentiate this with respect to y:

$$f_y = x\cdot\frac{1}{y} + g_y = \frac{x}{y} + g_y,$$

and this must equal $Q = \dfrac{x}{y} + \ln(5z)$. Hence $g_y = \ln(5z)$. Integrate with respect to y:

$$g(y,z) = \int \ln(5z)\,dy = y\ln(5z) + h(z),$$

where h is a function of z alone. Thus

$$f(x,y,z) = x\ln(2y) + y\ln(5z) + h(z).$$

Differentiate with respect to z:

$$f_z = y \cdot \frac{1}{z} + h'(z) = \frac{y}{z} + h'(z),$$

which must equal $R = \dfrac{y}{z}$. Therefore $h'(z) = 0$, so h is constant.

We may take the potential function (dropping the additive constant) as

$$\boxed{f(x, y, z) = x \ln(2y) + y \ln(5z).}$$

Conclusion: $\mathbf{F}$ is conservative on $D = \{y > 0,\ z > 0\}$, and one potential is $f(x, y, z) = x \ln(2y) + y \ln(5z)$.

Problem 10.

For a surface $z = f(x, y)$ the surface area over a region D is

$$A = \iint_D \sqrt{1 + f_x^2 + f_y^2}\, dA.$$

Here $f(x, y) = \dfrac{2}{3}(x^{3/2} + y^{3/2})$. Compute partial derivatives:

$$f_x = \frac{2}{3} \cdot \frac{3}{2} x^{1/2} = x^{1/2}, \qquad f_y = \frac{2}{3} \cdot \frac{3}{2} y^{1/2} = y^{1/2}.$$

Hence the integrand simplifies to

$$\sqrt{1 + f_x^2 + f_y^2} = \sqrt{1 + x + y}.$$

So the surface area is

$$A = \int_{y=0}^{1} \int_{x=0}^{2} \sqrt{1 + x + y}\, dx\, dy.$$

Evaluate the inner integral with respect to x. Using the antiderivative $\int \sqrt{a + x}\, dx = \frac{2}{3}(a + x)^{3/2}$,

$$\int_0^2 \sqrt{1 + x + y}\, dx = \frac{2}{3}\Big[(1 + 2 + y)^{3/2} - (1 + 0 + y)^{3/2}\Big] = \frac{2}{3}\Big[(3 + y)^{3/2} - (1 + y)^{3/2}\Big].$$

Thus

$$A = \frac{2}{3} \int_0^1 \Big[(3 + y)^{3/2} - (1 + y)^{3/2}\Big] dy.$$

Integrate termwise. Since $\int (a + y)^{3/2}\, dy = \frac{2}{5}(a + y)^{5/2}$, we get

$$A = \frac{2}{3} \cdot \frac{2}{5}\Big[(3 + y)^{5/2} - (1 + y)^{5/2}\Big]_0^1 = \frac{4}{15}\Big[(4)^{5/2} - (2)^{5/2} - (3)^{5/2} + (1)^{5/2}\Big].$$

To Compute the simple powers:

$$4^{5/2} = 32, \qquad 2^{5/2} = 4\sqrt{2}, \qquad 3^{5/2} = 9\sqrt{3}, \qquad 1^{5/2} = 1.$$

So

$$A = \frac{4}{15}\Big(32 - 4\sqrt{2} - 9\sqrt{3} + 1\Big) = \frac{4}{15}\Big(33 - 4\sqrt{2} - 9\sqrt{3}\Big).$$

You can leave the answer in this form or expand the numerator:

$$\boxed{A = \frac{132 - 16\sqrt{2} - 36\sqrt{3}}{15}}.$$

(Any algebraically equivalent expression is fine as the final answer.)

Problem 11.

Let $\mathbf{F} = (P, Q, R)$ with

$$P = -yx^2, \qquad Q = xy^2, \qquad R = e^{xy}.$$

Stokes' theorem says

$$\oint_C \mathbf{F} \cdot d\mathbf{r} = \iint_S (\nabla \times \mathbf{F}) \cdot \mathbf{n}\, dS,$$

where S is any oriented surface with boundary C and $\mathbf{n}$ is the unit normal consistent with the orientation of C. Take S to be the disk in the xy-plane of radius 3 and $\mathbf{n} = \mathbf{k} = (0, 0, 1)$ since C is counterclockwise viewed from above.

Compute the curl:

$$\nabla \times \mathbf{F} = \Big(\frac{\partial R}{\partial y} - \frac{\partial Q}{\partial z}, \frac{\partial P}{\partial z} - \frac{\partial R}{\partial x}, \frac{\partial Q}{\partial x} - \frac{\partial P}{\partial y}\Big).$$

Evaluate each component:

$$\frac{\partial R}{\partial y} = \frac{\partial}{\partial y} e^{xy} = xe^{xy}, \qquad \frac{\partial Q}{\partial z} = 0,$$

$$\frac{\partial P}{\partial z} = 0, \qquad \frac{\partial R}{\partial x} = \frac{\partial}{\partial x} e^{xy} = ye^{xy},$$

$$\frac{\partial Q}{\partial x} = \frac{\partial}{\partial x}(xy^2) = y^2, \qquad \frac{\partial P}{\partial y} = \frac{\partial}{\partial y}(-yx^2) = -x^2.$$

Hence

$$\nabla \times \mathbf{F} = (xe^{xy},\ -ye^{xy},\ y^2 + x^2).$$

Dot with the unit normal $\mathbf{k}$:

$$(\nabla \times \mathbf{F}) \cdot \mathbf{k} = x^2 + y^2.$$

Therefore,

$$\oint_C \mathbf{F} \cdot d\mathbf{r} = \iint_D (x^2 + y^2)\, dA,$$

where D is the disk $x^2 + y^2 \le 9$. Switch to polar coordinates $x = r\cos\theta,\ y = r\sin\theta$, $dA = r\, dr\, d\theta$. Then

$$\iint_D (x^2 + y^2)\, dA = \int_{\theta=0}^{2\pi} \int_{r=0}^{3} r^2 \cdot r\, dr\, d\theta = \int_0^{2\pi} \int_0^3 r^3\, dr\, d\theta.$$

Compute the radial integral:

$$\int_0^3 r^3\, dr = \frac{r^4}{4}\bigg|_0^3 = \frac{81}{4}.$$

Thus

$$\oint_C \mathbf{F} \cdot d\mathbf{r} = \int_0^{2\pi} \frac{81}{4}\, d\theta = \frac{81}{4} \cdot 2\pi = \frac{81\pi}{2}.$$

$$\boxed{\oint_C \mathbf{F} \cdot d\mathbf{r} = \frac{81\pi}{2}}.$$

Problem 12.

By the Divergence Theorem,

$$\iint_S \mathbf{F} \cdot d\mathbf{S} = \iiint_V \nabla \cdot \mathbf{F}\, dV,$$

where V is the solid ball $x^2 + y^2 + z^2 \le 3^2$.

Compute the divergence. Write $\mathbf{F} = (P, Q, R)$ with

$$P = 2x^3 + 2y^3, \qquad Q = 2y^3 + 2z^3, \qquad R = 2z^3 + 2x^3.$$

Then

$$\frac{\partial P}{\partial x} = 6x^2, \quad \frac{\partial Q}{\partial y} = 6y^2, \quad \frac{\partial R}{\partial z} = 6z^2,$$

so

$$\nabla \cdot \mathbf{F} = 6(x^2 + y^2 + z^2) = 6r^2.$$

Switch to spherical coordinates (r, θ, ϕ) with $dV = r^2 \sin\phi\, dr\, d\phi\, d\theta$. Thus, the flux is

$$\iiint_V 6r^2\, dV = 6 \int_{\theta=0}^{2\pi} \int_{\phi=0}^{\pi} \int_{r=0}^{3} r^2 \cdot r^2 \sin\phi\, dr\, d\phi\, d\theta = 6 \left(\int_0^3 r^4\, dr \right) \left(\int_0^{\pi} \sin\phi\, d\phi \right) \left(\int_0^{2\pi} d\theta \right).$$

Evaluate the factors:

$$\int_0^3 r^4\, dr = \frac{3^5}{5} = \frac{243}{5}, \qquad \int_0^{\pi} \sin\phi\, d\phi = 2, \qquad \int_0^{2\pi} d\theta = 2\pi.$$

Therefore,

$$\iint_S \mathbf{F} \cdot d\mathbf{S} = 6 \cdot \frac{243}{5} \cdot 2 \cdot 2\pi = \frac{6 \cdot 243 \cdot 4\pi}{5} = \frac{5832\pi}{5}.$$

$$\boxed{\iint_S \mathbf{F} \cdot d\mathbf{S} = \frac{5832\pi}{5}}.$$